A LAYER STRIPPING ALGORITHM IN ELASTIC IMPEDANCE TOMOGRAPHY

GEN NAKAMURA* AND GUNTHER UHLMANN†

1. Introduction. Let $n \geq 2$ be an integer and $\Omega \subset \mathbb{R}^n$ be a bounded domain with smooth boundary $\Gamma = \partial\Omega$. We consider Ω as a reference configuration domain of an elastic medium with elastic tensor $C(x) = (C_{ijkl}(x))_{1\leq i,j,k,l\leq n} \in C^\infty(\overline{\Omega})$. We assume that $C(x)$ satisfies the following hyperelasticity and strong convexity conditions:
(Hyperelasticity)

$$C_{ijkl}(x) = C_{klij}(x), \quad \forall x \in \overline{\Omega}, \quad 1 \leq i,j,k,l \leq n. \tag{1.1}$$

(Strong Convexity) There exists $\delta > 0$ such that for any $x \in \overline{\Omega}$ and real matrix (ϵ_{ij}),

$$\sum_{i,j,k,l=1}^{n} C_{ijkl}(x)\epsilon_{ij}\epsilon_{kl} \geq \delta \sum_{i,j=1}^{n} \epsilon_{ij}^2. \tag{1.2}$$

The problem of *Elastic Impedance Tomography* consists in determining the elastic tensor C by making displacement and traction measurements at the boundary of the domain. This information is encoded in the so-called Dirichlet to Neumann map. Considerably progress has been made in recent years in the question of identifiability in the case that the elastic tensor is isotropic ([N-U I,II]). A key ingredient in the global identifiability result of [N-U I] is the construction of complex geometrical optics (or exponentially growing solutions) for the Lamé system and in fact for any differential system that can be reduced to a first order perturbation of the Laplacian. For other applications of this construction to other inverse boundary value problems involving first order perturbations of the Laplacian see [U].

In this paper we develop a layer stripping algorithm for isotropic elastic materials in all dimensions $n \geq 2$ and for a large class of anisotropic elastic materials in three dimensions, the so-called transversally isotropic materials. We describe below in more detail the mathematical problem and our results.

Let $0 < t \ll 1$ and define $\Omega(t) := \{x \in \Omega; \mathrm{dist}(x, \partial\Omega) > t\}$. Then the boundary $\Gamma(t)$ of $\Omega(t)$ is smooth.

Let us consider the boundary value problem $(BP)_t$:

$$(BP)_t \begin{cases} (Lu)_i(x) := \displaystyle\sum_{j,k,l=1}^{n} \partial_{x_j}(C_{ijkl}(x)\partial_{x_l}u_k(x)) = 0 & \text{in } \Omega(t) \\ u_i = f_i \in C^\infty(\Gamma(t)) & \text{on } \Gamma(t) \quad (1 \leq i \leq n). \end{cases}$$

* Department of Mathematics, Science University of Tokyo, Tokyo, 162, Japan.
† Department of Mathematics, University of Washington, Seattle, WA 98195, USA.
Partly supported by NSF Grant DMS–9322619 and ONR grant N0014–93–1–0295.

G. Chavent et al. (eds.), *Inverse Problems in Wave Propagation*

where $u(x) = (u_1(x), \ldots, u_n(x))$ is the displacement vector of Ω. It is well known that $(BP)_t$ is well posed.

DEFINITION 1.1. *We define the Dirichlet to Neumann map (DN) in* $\Omega(t)$, $\Lambda(t)$, *by*

$$(1.3) \qquad (\Lambda(t)f)_i = \sum_{j,k,l=1}^{n} \nu_j C_{ijkl} \partial_{x_l} u_k|_{\Gamma(t)}, \quad i = 1, \ldots, n$$

where $u = (u_1, \ldots, u_n) \in C^\infty(\overline{\Omega(t)})$ *is a solution of* $(BP)_t$, $f = (f_1, \ldots, f_n) \in C^\infty(\partial\Omega)$ *and* $\nu = (\nu_1, \ldots, \nu_n)$ *is the outer unit normal of* $\Gamma(t)$.

The layer stripping algorithm consists in finding an approximation for the medium parameters layer by layer from the Dirichlet to Neumann map $\Lambda(0)$. This consists in two steps:
1) First one finds the parameters of the medium at the boundary $\Gamma(t)$ from $\Lambda(t)$.
2) One derives a differential equation for $\Lambda(t)$ in $\Omega(t)$ involving $\Lambda(t)$ and tangential derivatives of the medium parameters that allows to approximately propagate the boundary data to the interior, layer by layer. by using the approximation

$$\Lambda(t + \delta t) - \Lambda(t) \sim \frac{d\Lambda(t)}{dt} \delta t.$$

This algorithm has been developed for different inverse problems (see for instance the Proceedings [C] and the references there). For the electrical impedance imaging problem it was analyzed in detail in [S-C-I-I]. A convergent layer stripping algorithm was developed in [S] in two dimensions for the case in which the conductivity depends only on the radius.

In this paper we derive a Riccati equation for $\Lambda(t)$ for any anisotropic elastic medium satisfying (1.1) and (1.2). We also prove that $\Lambda(t)$ is a classical pseudodifferential operator of order 1 on $\Gamma(t)$ and that the full symbol of this operator determines the full Taylor series of the surface impedance tensor that we describe below. From the surface impedance tensor it is known that we can recover the Taylor series at the boundary of the Lamé parameters for isotropic medium ([N-U]) in dimension $n \geq 2$ and the Taylor series at the boundary of the elastic tensor for a class of anisotropic medium in two dimensions ([N-U]) and for transversally isotropic materials in three dimensions ([N-T].) Thus a layer stripping algorithm is derived for these kind of elastic materials. The general question of under which conditions one can determine the boundary values of the elastic tensor for general anisotropic materials is open. It is known that this is not true in general ([N-T I]).

We take boundary normal coordinates $(x^1, \ldots, x^{n-1}, x^n) = (y, x^n)$ such that $\Gamma(t)$ is locally expressed as $x^n = t$. Let $\sigma(\Lambda(t))(y, \eta)$ and $\tilde{\sigma}(\Lambda(t))(y, \eta)$ be the principal symbol and the full symbol respectively of the pseudodifferential operator $\Lambda(t)$. The tensor

$$Z_t(y,\eta) := |\eta|^{-1}\sigma(\Lambda(t))(y,\eta)$$

is called the surface impedance tensor in Stroh's formalism for anisotropic elasticity (see [C-S]).

In this paper we will prove that $\widetilde{\sigma}(\Lambda(t))(y,\eta)$ determines $\Phi(t) := \left\{ \left(\frac{d}{ds}\right)^j Z_s(y,\eta)\Big|_{s=t}; j \in \mathbb{Z}_+ := \mathbb{N} \cup \{0\} \right\}$.

The method of proof of this result follows the ideas in [L-U] for the anisotropic electrical impedance tomography. In [L-U] a factorization of the conductivity equation into a heat equation and a backwards heat equation is used to prove the boundary determination of anisotropic conductivities in special coordinates. This method has the following advantages: a) It derives the Riccati equation for the DN map. b) It proves that the DN map is a pseudodifferential operator. Then, using a) and the calculus of pseudodifferential operators, one can compute its full symbol which contains information of the medium parameters at the boundary. A Riccati equation for the DN map associated to anisotropic conductivities was derived in [L-U]. From the full symbol of the DN map one can recover the full Taylor series of the conductivity at the boundary ([L-U]). However this was not realized as a layer stripping algorithm.

In this paper we develop this approach for the considerable more complicated case of anisotropic elastic materials. In section 2 we develop the factorization of the elasticity operator and derive the Riccati equation for the DN map. In section 3 we show that the full symbol of the DN map determines $\Phi(t)$. Therefore combining this with the boundary determination results in [N-U] and [N-T] we have completed the layer stripping algorithm for isotropic and transversally isotropic materials. We expect that this will be extended to other type of anisotropic materials.

2. Factorization of L. Let $(x^1,\dots,x^{n-1},x^n) = (y,x^n)$ be boundary normal coordinates as previously defined. Then the tensorial local expression of $(BP)_t$ in terms of these coordinates is

$$(BP)_t \begin{cases} (Lu)^i = \displaystyle\sum_{j,k,l=1}^{n} \nabla_j(C^{ijkl}\nabla_l u_k) = 0 & \text{in}\{x^n > t\} \\ u^i\big|_{x^n=t} = f^i & (1 \le i \le n) \end{cases}$$

where ∇_j is the covariant derivative with respect to $\frac{\partial}{\partial x^j}$ and

$$C^{ijkl}(x) = \sum_{a,b,c,d=1}^{n} g^{ai}(x)g^{bj}(x)g^{ck}(x)g^{dl}(x)C_{abcd}(x).$$

where

$$g^{ai}(x) = \sum_{r=1}^{n} \frac{\partial x^a}{\partial x_r}(x)\frac{\partial x^i}{\partial x_r}(x).$$

We define

$$
(2.1)\quad
\begin{aligned}
Q(x,\eta) &:= \left(\sum_{j,l=1}^{n-1} C^{ijkl}(x)\eta_j\eta_l;\ \begin{matrix} i\downarrow 1,\dots,n \\ k\rightarrow 1,\dots,n\end{matrix}\right)\\
A_1^h(x,\eta) &:= R(x,\eta)+{}^tR(x,\eta)\\
R(x,\eta) &:= \left(\sum_{j=1}^{n-1} C^{ijkn}(x)\eta_j;\ \begin{matrix} i\downarrow 1,\dots,n \\ k\rightarrow 1,\dots,n\end{matrix}\right)\\
T(x) &:= \left(C^{inkn}(x);\ \begin{matrix} i\downarrow 1,\dots,n \\ k\rightarrow 1,\dots,n\end{matrix}\right)
\end{aligned}
$$

The principal symbol of $-L$ is given by

$$
(2.2)\qquad M(x,\eta,\xi_n) := T(x)\xi_n^2 + A_1^k(x,\eta)\xi_n + Q(x,\eta)
$$

where η denotes the dual variable to y and ξ_n denotes the dual variable to x^n. For the operators we have

$$
(2.3)\quad -L = T(x)\widetilde{D}_{x^n}^2 + A_1^h(x,\widetilde{D}_y)\widetilde{D}_{x^n} + Q(x,\widetilde{D}_y) + F_0(x)\widetilde{D}_{x^n} + F_1(x,\widetilde{D}_y)
$$

where $\sqrt{-1}F_0(x)$ is a real-valued matrix and $F_1(x, D_{\tilde{y}})$ is a first order differential system, where

$$
\begin{gathered}
\widetilde{D}_{x^n} = -\sqrt{-1}\nabla_n\\
\widetilde{D}_y = (\widetilde{D}_{y^0},\dots,\widetilde{D}_{y^{n-1}})\\
\widetilde{D}_{y^j} = -\sqrt{-1}\nabla_j
\end{gathered}
$$

Using the hyperelasticity condition (1.1) we have that $Q(x,\eta)$ and $T(x)$ are symmetric $n\times n$ matrices and by the strong convexity condition (1.2), $T(x)$ and $M(x,\eta,\xi_n)$ are positive definite matrices for $x\in\overline{\Omega}$, $(\eta,\xi_n)\in\mathbb{R}^n\backslash 0$. Hence for fixed (x,η), $M(x,\eta,\xi_n)=0$ in ξ_n admits n number of roots $\xi_n=\zeta_j$ $(j=1,\dots,n)$ with positive imaginary parts and n number of roots $\xi_n=\overline{\zeta_j}$ $(j=1,\dots,n)$. The following result gives the factorization of the principal symbol of $-L$.

THEOREM 2.1. ([G-L-R] *Let*

$$
(2.4)\qquad B_1(x,\eta) := \left(\oint_\gamma \zeta M(x,\eta,\zeta)^{-1}d\zeta\right)\left(\oint_\gamma M(x,\eta,\zeta)^{-1}d\zeta\right)^{-1}
$$

where $\gamma\subset\mathbb{C}_+ := \{\zeta\in\mathbb{C};\operatorname{Im}\zeta>0\}$ *is a closed contour enclosing all the* ζ_j $(j=1,\dots,n)$. *Then we have*

$$
(2.5)\qquad M(x,\eta,\xi_n) = (\xi_n - B_1^*(x,\eta))T(x)(\xi_n - B_1(x,\eta))
$$

with $\mathrm{Spec}(B_1(x,\eta)) \subset \mathbb{C}_+$ *where* $\mathrm{Spec}(B_1(x,\eta)) := \{\text{spectrum of } B_1(x,\eta)\}$.

THEOREM 2.2. *There exist classical pseudodifferential operators* $B(x, D_y)$ *and* $G_0(x, D_y)$ *of orders 1 and 0 respectively depending smoothly on* x^n $(0 \le x^n \ll 1)$ *such that*

$$\sigma(B)(x,\eta) = B_1(x,\eta) \tag{2.6}$$

and

$$-L = (\tilde{D}_{x^n} - B^*(x,\tilde{D}_y) + G_0(x,\tilde{D}_y))T(x)(\tilde{D}_{x^n} - B(x,\tilde{D}_y)). \tag{2.7}$$

Moreover the operator $B = B(x, D_y)$ *satisfies the Riccati equation*

$$\tilde{D}_{x^n}B + (T^{-1}A_1^h + T^{-1}F_0)B + B^2 + T^{-1}Q + T^{-1}F_1 = 0. \tag{2.8}$$

and

$$\tilde{D}_{x^n}u|_{x^n=t} = B(x^n, y, \tilde{D}_y)|_{x^n=t}(f) + K(f) \tag{2.9}$$

where K *is a smoothing operator and* u *is a solution of* $(BP)_t$.

Remark. We can more conveniently rewrite (2.8) in the form:

$$\tilde{D}_{x^n}B+T^{-1}\{(TB+B^*T+A_1^h)+F_0\}B+T^{-1}\{(Q-B^*TB)+F_1\} = 0 \tag{2.10}$$

Observe that each of the summands of (2.10) have order 1 since $Q - B^*TB$ is of order one by (2.2) and (2.5).

COROLLARY 2.1. *Let us consider* $\Lambda(t)$ *as an operator sending covariant vector functions to covariant vector functions. We define*

$$\hat{\Lambda}(t) := \sqrt{-1}T^{-1}g^*\Lambda(t), \tag{2.11}$$

that is

$$(T^{-1}g^*\Lambda(t)f)_i = \sum_{k,l,m=1}^{n} (T^{-1})_{ik}\ g^{kl}\Lambda_l^m f_m, \quad i = 1,\dots,n$$

where

$$\Lambda(t) = (\Lambda_i^j(t))_{1\le i,j\le n}, \quad f = (f_1,\dots,f_n)$$

and

$$g^*(x) := (g^{ij})(x) = \sum_{k=1}^{n} \frac{\partial x^i}{\partial x_k}(x)\frac{\partial x^j}{\partial x_k}(x). \tag{2.12}$$

Then $\hat{\Lambda}(t)$ *satisfies, modulo a smoothing operator, the Riccati equation*

$$D_t\hat{\Lambda}(t)+J_1(t)\hat{\Lambda}(t)+\hat{\Lambda}(t)K_1(t)+\hat{\Lambda}(t)^2+F_2(t) = 0 \quad (0 \le t \ll 1) \tag{2.13}$$

where

$$J_1(t) := T^{-1}(A_1^h + B_0 - E_1)\big|_{x^n=t} \tag{2.14}$$

with

$$E_1 := (A_1^h - R)(x, \tilde{D}_y),$$

where

$$K_1(t) := -T^{-1}E_1\big|_{x^n=t}, \tag{2.15}$$

and

$$\begin{aligned} F_2(t) \;:=\; &\{T^{-1}(Q + F_1) - \tilde{D}_{x^n}(T^{-1}E_1) \\ &-T^{-1}(A_1^h + F_0)T^{-1}E_1 + (T^{-1}E_1)^2\}\big|_{x^n=t} \end{aligned} \tag{2.16}$$

Proof of Corollary 2.1. This easily follows from the fact that

$$g^*\Lambda(t) = -\sqrt{-1}(TB + E_1)\big|_{x^n=t} \quad \text{mod smoothing} \tag{2.17}$$

and (2.7).

Proof of Theorem 2.2. From (2.7), B and G_0 must satisfy

$$-TB + \tilde{D}_{x^n}T - B^*T + G_0T = A_1^h + F_0 \tag{2.18}$$

$$\begin{aligned} &-T\tilde{D}_{x^n}B - (\tilde{D}_{x^n}T)B + B^*TB - G_0TB = Q + F_1 \\ &\text{where} \quad A_1^h = A_1^h(x, \tilde{D}_y), Q = Q(x, \tilde{D}_y), F_1 = F_1(x, \tilde{D}_y). \end{aligned} \tag{2.19}$$

By eliminating G_0 from (2.19) using (2.18), we get (2.8).

Now by the composition formula for pseudodifferential operator, we have from (2.8),

$$\begin{aligned} \sum_\alpha(\alpha!)^{-1}\partial_\eta^\alpha\tilde{\sigma}(B)D_y^\alpha\tilde{\sigma}(B) + \sum_\alpha(\alpha!)^{-1}\partial_\eta^\alpha(T^{-1}A_1^h)D_y^\alpha\tilde{\sigma}(B) + T^{-1}Q + D_{x^n}\tilde{\sigma}(B) + \\ + T^{-1}F_0\tilde{\sigma}(B) + T^{-1}F_1 \sim 0. \end{aligned} \tag{2.20}$$

where (2.20) is interpreted in the asymptotic sense of symbols. If we substitute

$$\tilde{\sigma}(B)(x,\eta) \sim \sum_{j=0}^{\infty} B_{1-j}(x,\eta) \tag{2.21}$$

with each $B_{1-j}(x,\eta)$ homogeneous of degree $1-j$ with respect to η for $|\eta| \geq 1$ into (2.20), we obtain the following conditions. Grouping the homogeneous terms of degree 2 we have

$$B_1^2 + T^{-1}\sigma(A_1^h)B_1 + T^{-1}\sigma(Q) = 0 \tag{2.22}$$

which is valid by (2.22). Grouping the homogeneous terms of degree 1 we have

$$B_0B_1 + B_1B_0 + T^{-1}\sigma(A_1^h)B_0 = \sum_{j=1}^{n-1}\partial_{\eta_j}B_1D_{y^j}B_1 - \sum_{j=1}^{n-1}\partial_{\eta_j}(T^{-1}\sigma(A_1^h))D_{y^j}B_1 - D_{x^n}B_1 - T^{-1}F_0B_1 - T^{-1}\sigma(F_1) = 0. \tag{2.23}$$

and grouping the terms homogeneous of degree 0 we have

$$\begin{aligned} B_{-1}B_1 + B_1B_{-1} + T^{-1}\sigma(A_1^h)B_{-1} = &-\sum_{j=1}^{n-1}(\partial_{\eta_j}BD_{y^j}B_1 + \partial_{\eta_j}B_1D_{y^j}B_0) \\ &- \sum_{|\alpha|=2}(\alpha!)^{-1}\partial_\eta^\alpha B_1D_y^\alpha B_1 \\ &-\sum_{j=1}^{n-1}\partial_{\eta_j}(T^{-1}\sigma(A_1^h))D_{y^j}B_0 - D_{x^n}B_0 - T^{-1}F_0B_0 - T^{-1}\sigma_0(F_1) \end{aligned} \tag{2.24}$$

where $\sigma_0(F_1)$ is the term homogeneous of degree 0 in $\tilde{\sigma}(F_1)$. Moreover grouping the homogeneous terms of degree $-r$ ($r \in \mathbb{N}$) we have

$$\begin{aligned} B_{-r-1}B_1 + B_1B_{-r-1} + T^{-1}\sigma(A_1^h)B_{-r-1} = &- \sum_{\substack{j+k=r+2 \\ j,k\geq 1}} B_{1-j}B_{1-k} \\ &- \sum_{\substack{|\alpha|\geq 1 \\ j+k=r-|\alpha|+2 \\ j,k\geq 0}} (\alpha!)^{-1}\partial_\eta^\alpha B_{1-j}D_y^\alpha B_{1-k} \\ &-\sum_{j=1}^{n-1}\partial_{\eta_j}(T^{-1}\sigma(A_1^h))D_{y^j}B_{-r} - D_{x^n}B_{-r} - T^{-1}F_0B_{-r}. \end{aligned} \tag{2.25}$$

To see that (2.22)–(2.25) are solvable, we note that we have from (2.5),

$$\begin{aligned} B_{-r}B_1 + B_1B_{-r} + T^{-1}\sigma(A_1^h)B_{-r} = \\ (-T^{-1}\sigma(Q)B_1^{-1})B_{-r} - B_{-r}(-B_1) \qquad (r \in \mathbb{Z}_+), \end{aligned} \tag{2.26}$$

Also from the fact that $\mathrm{Spec}(B_1) \subset \mathbb{C}_+$ and the strong convexity condition (1.2) we conclude

$$\mathrm{Spec}(-B_1) \subset \mathbb{C}_-, \qquad \mathrm{Spec}(-T^{-1}\sigma(Q)B_1^{-1}) \subset \mathbb{C}_+. \tag{2.27}$$

Once B is determined modulo a smoothing operator we can determine G_0 modulo a smoothing operator from (2.18). The recursion formula for $G_{0,-j}$ ($j \in \mathbb{Z}_+$) with

$$\tilde{\sigma}(G_0) \sim \sum_{j=0}^{\infty} G_0, -j \tag{2.28}$$

is given as follows:

$$(2.29)\quad G_{0,0} = \sum_{j=1}^{n-1} T\partial_{\eta_j} B_1 D_{y^j} T^{-1} - (D_{x^n}T)T^{-1} + \sum_{j=1}^{n-1} \partial_{\eta_j} D_{y^j} B_1^* + B_0 T^{-1}$$

$$G_{0,-j} = \sum_{|\alpha|\le j+1} (\alpha!)^{-1}(T\partial_\eta^\alpha B_{-j+|\alpha|} D_y^\alpha T^{-1} + \partial_\eta^\alpha D_y^\alpha B^*_{-j+|\alpha|}) \quad (j \in \mathbb{N}).$$
(2.30)

Now using the factorization (2.7) in a backwards heat equation and a heat equation we can argue as in [L-U], Proposition 1.2 to conclude (2.9) finishing the proof of the Theorem. Since we can see from (2.7), (2.17) and $\mathrm{Spec}(B_1^*) \subset \mathbb{C}_-$ that $\Lambda(t)$ is a classical pseudodifferential operator of order 1 depending smoothly on t $(0 \le t \ll 1)$.

3. The identification of $\Phi(t)$ from $\tilde{\sigma}(\Lambda(t))$. Let

$$(3.1)\qquad g^*\Lambda(t) \sim \sum_{j=1}^{\infty} \lambda_{1-j}(t)$$

where each $\lambda_{1-j}(t)(y,\eta)$ is homogeneous of degree $1-j$ with respect to η for $|\eta| \ge 1$. For $k \in \mathbb{Z}_+$, $l \in \mathbb{R}$, mod $(\tilde{T}_t^k, S^l)$ means we are neglecting the terms in $S^l :=$ {the symbol $p_t(y,\eta)$ of a classical pseudodifferential operator $p_t(y, D_y)$ of order l depending smoothly on t $(0 \le t \ll 1)$} and the term $\notin S^l$ which depends only on the t-derivatives of $C(y,t)$ up to order k.

THEOREM 3.1. *There is a linear bijective map $W(x,\eta)$ on the set of all $n \times n$ matrices which depend only on $C(x)$ but not on its derivatives such that*

$$\lambda_{-r}(t)(y,\eta) = |\eta|^{-1} W(y,t,\eta)(D_t\lambda_{1-r}(t)(y,\eta)) \ \mathit{mod}\ (\tilde{T}_t^r, S^{-r-1}) \quad (r \in \mathbb{Z}_+).$$
(3.2)

Therefore $\tilde{\sigma}(\Lambda(t))(y,\eta)$ determines $\Phi(t)$.

Proof. By (2.17) we have

$$(3.3)\qquad \lambda_{1-j}(t)(y,\eta) = -\sqrt{-1}TB_{1-j}(x,\eta)\big|_{x^n=t} \quad (j \in \mathbb{N}).$$

We prove (3.2) by induction on r. From (2.23)

$$B_0B_1 + B_1B_0 + T^{-1}\sigma(A_1^h)B_0 = -D_{x^n}B_1 - T^1F_0B_1 - T^{-1}F_1 \ \mathrm{mod}\ (\tilde{T}^0_{x^n}, S^0)$$

and from (2.7)

$$\begin{aligned} F_0 &= D_{x^n}T \ \mathrm{mod}\ (\tilde{T}^0_{x^n}, S^{-1}) \\ F_1 &= D_{x^n}E_1 \ \mathrm{mod}\ (\tilde{T}^0_{x^n}, S^0). \end{aligned}$$

Hence

$$\begin{aligned} B_0B_1 + B_1B_0 + T^{-1}\sigma(A_1^h)B_0 &= -D_{x^n}B_1 - T^{-1}(D_{x^n}T)B_1 - T^{-1}D_{x^n}E_1 \\ &= -T^{-1}D_{x^n}(TB_1 + E_1) \\ &= -\sqrt{-1}T^{-1}D_{x^n}\lambda_1(x^n) \bmod (\widetilde{T}^0_{x^n}, S^0). \end{aligned}$$

Using (3.3) we have

$$\begin{aligned} B_0B_1 + B_1B_0 + T^{-1}\sigma(A_1^h)B_0 &= (-T^{-1}\sigma(Q)B_1^{-1})B_0 - B_0(-B_1) \\ &= \sqrt{-1}T^{-1}\{(-\sigma(Q)B_1^{-1}T^{-1})(-\sqrt{-1}TB_0) \\ &\qquad -(-\sqrt{-1}TB_0)(-B_1\} \\ &= \sqrt{-1}T^{-1}\{(-\sigma(Q)B_1^{-1}T^{-1})\lambda_0(x^n) - \lambda_0(x^n)(-B_1)\} \end{aligned}$$

Here note that

$$\text{(3.4)} \qquad \mathrm{Spec}(-B_1) \subset \mathbb{C}_-, \qquad \mathrm{Spec}(-\sigma(Q)B_1^{-1}T^{-1}) \subset \mathbb{C}_+$$

So if we denote by $W(x)(Y)$ the solution X of the matrix equation

$$X(-|\eta|^{-1}B_1) - (-|\eta|^{-1}\sigma(Q)B_1^{-1}T^{-1})X = Y,$$

we have

$$\text{(3.5)} \qquad \lambda_0(t) = |\eta|^{-1}W(y,t,\eta)(D_t\lambda_1(t)) \bmod (\widetilde{T}^0_t, S^{-1})$$

Hence (3.2) holds for $r = 0$.

Now let $q \in \mathbb{N}$ and let us assume (3.2) is true for all $r \leq q$. From (3.2) and (3.3) we have that each B_{1-j} $(j \leq q+1)$ depends only on $C(x)$ and its x^n-derivatives of $C(x)$ up to order j. Hence from (2.25)

$$\begin{aligned} &\sqrt{-1}T^{-1}\{(-\sigma(Q)B_1^{-1}T^{-1})\lambda_{-q-1}(x^n) - \lambda_{-q-1}(x^n)(-B_1)\} \\ &\quad = -\sqrt{-1}T^{-1}D_{x^n}\lambda_{-q}(x^n) \bmod (\widetilde{T}^{q+1}_{x^n}, S^{-q-1}). \end{aligned}$$

By the definition of $W(x)$ we have

$$\lambda_{-q-1}(t) = |\eta|^{-1}W(D_t\lambda_{-q}(t)) \bmod (\widetilde{T}^{q+1}_{x^n}, S^{-q-2})$$

REFERENCES

[C] *Invariant imbedding and inverse problems*, SIAM Proceedings series list (1991) edited by J.P. Corones, G. Kristensson, P.Nelson and D.L. Seth.

[C-S] Chadwick, P. and Smith, G.D., Foundations of the theory of surface waves in anisotropic elastic materials, *Adv. Appl. Mech.*, **17**, (1977), 303–376.

[G-L-R] Gohberg, I., Lancaster, P. and Rodman, L., *Matrix Polynomial*, Academic Press, 1982.

[L-U] Lee, J. and Uhlmann, G., Determining anisotropic real-analytic by boundary conductivity measurements, *CPAM*, **42**, 1989, pp. 1097–1112.

[N-U] Nakamura, G., and Uhlmann, G., Inverse problems at the boundary for an elastic medium, *SIAM J. Math. Anal.*, **26**, (1995), pp. 263–279.

[N-U I] Nakamura, G., and Uhlmann, G., Global uniqueness for an inverse boundary value problem arising in elasticity, *Invent. Math.*, **118**, (1994), 457–474.

[N-U II] Nakamura, G., and Uhlmann, G., Identification of Lamé parameters by boundary measurements, *American Journal of Math.*, **115**, (1993), 1161–1187.

[N-T] Nakamura, G., Tanuma, K., Inverse problems at the boundary for a transversally isotropic elastic medium in three dimensional space (in preparation).

[N-T I] Nakamura, G., Tanuma, K., Non uniqueness theorem for an inverse problem at the boundary, to appear SIAM J. Appl. Math.

[S-C-I-I] Somersalo, E., Cheney, M., Isaacson, D., and Isaacson E.L., Layer stripping: A direct numerical method for impedance imaging, *Inverse Problems* **7**, (1991), 899-926.

[S] Sylvester, J., A convergent layer stripping algorithm for the radially symmetric impedence tomography problem, *Comm. PDE*, **17** (1992), 1955–1994.

[U] Uhlmann, G., Inverse boundary value problems for first order perturbations of the Laplacian, *Lectures in Appl. Math.*, **30**, (1994), 245–258.